1 Farey Sequences

Definition 1 Farey Sequence. The Farey sequence of order $n$, denoted $F_n$, is the sequence of completely reduced fractions between 0 and 1 which, in lowest terms, have denominators less than or equal to $n$, arranged in order of increasing size.

Example 1

$F_1 = \{0/1, 1/1\}$
$F_2 = \{0/1, 1/2, 1/1\}$
$F_3 = \{0/1, 1/3, 1/2, 2/3, 1/1\}$
$F_4 = \{0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1\}$
$F_5 = \{0/1, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1\}$
$F_6 = \{0/1, 1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 1/1\}$
$F_7 = \{0/1, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 2/5, 3/7, 1/2, 4/7, 3/5, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 1/1\}$

Properties of Farey Sequences.

- $F_n$ contains $F_k$ for all $k \leq n$.
- $F_n$ is equal to $F_{n-1}$ plus an additional fraction for each number that is less than $n$ and coprime to $n$. For example, $F_6$ consists of $F_5$ together with $1/6$ and $5/6$.
- Let $|F_n|$ denote the number of fractions in $F_n$. For $n > 1$, $|F_n|$ is odd and the middle term of $F_n$ is equal to $1/2$.
- $|F_n| = |F_{n-1}| + \phi(n)$
- Since $|F_1| = 2$, we obtain

$$|F_n| = 1 + \sum_{k=1}^{n} \phi(k),$$

where $\phi(k)$ is Euler’s totient function ($\phi(k)$ is equal to the number of positive integers less than or equal to $k$ that are relatively prime to $k$).

- Example of the mediant property. Unfortunately, addition of fractions is not as easy as we would like it to be. For example,

$$\frac{1}{5} + \frac{1}{3} \neq \frac{1+1}{5+3} = \frac{1}{4}.$$


But, looking at the Farey sequences, how does $1/4$ relate to $1/5$ and $1/3$? Repeat for additional consecutive terms.

- **Further example of the mediant property.** Choose 3 consecutive terms of $F_n$, say $p_1/q_1, p_2/q_2, p_3/q_3$. Compute

$$\frac{p_1 + p_3}{q_1 + q_3}.$$ 

What do you observe?

- **The mediant property.** How do we go from the $(n - 1)$-st row to the $n$-th row? Show that if $0 < a/b < c/d < 1$, then

$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}.$$ 

Thus we have the following algorithm:

**Algorithm 1 How to Compute $F_n$.**

1. Copy $F_{n-1}$ in order.
2. Insert the mediant fraction $\frac{a + c}{b + d}$ between $\frac{a}{b}$ and $\frac{c}{d}$ if $b + d \leq n$. (If $b + d > n$, the mediant $\frac{a + c}{b + d}$ will appear in a later sequence).

Use this algorithm to find $F_4$ from $F_3$. Then find $F_5$.

- Choose 2 consecutive terms of $F_n$, say $p_1/q_1$ and $p_2/q_2$. Compute $p_2/q_2 - p_1/q_1$. Compute $p_2q_1 - p_1q_2$. What do you observe? Make and prove a conjecture.

- Suppose that $p_1/q_1$ and $p_2/q_2$ are two successive terms of $F_n$. Prove that $p_2q_1 - p_1q_2 = 1$. Note that it is equivalent to prove that if $p_1/q_1$ and $p_2/q_2$ are two successive terms of $F_n$ with $p_1/q_1$ less than $p_2/q_2$, then

$$\frac{p_2}{q_2} - \frac{p_1}{q_1} = \frac{1}{q_1q_2}.$$ 

Use induction on $n$ and the mediant property to prove this result.

- Prove that if $p_1/q_1$, $p_2/q_2$, and $p_3/q_3$ are three successive terms of $F_n$, then

$$\frac{p_2}{q_2} = \frac{p_1 + p_3}{q_1 + q_3}.$$
2 Ford Circles

Definition 2 Ford Circle. For every rational number \( p/q \) in lowest terms, the Ford circle \( C(p, q) \) is the circle with center \( \left( \frac{p}{q}, \frac{1}{2q^2} \right) \) and radius \( \frac{1}{2q^2} \). This means that \( C(p, q) \) is the circle tangent to the \( x \)-axis at \( x = p/q \) with radius \( \frac{1}{2q^2} \). Observe that every small interval of the \( x \)-axis contains points of tangency of infinitely many Ford circles.

Example 2 Sketch \( C(0,1) \), \( C(1,1) \), \( C(1,2) \), \( C(1,3) \), \( C(2,3) \).

Example 3 Consider three adjacent terms of \( F_n \). What do you observe about the corresponding Ford circles?

Theorem 1 No Ford circles intersect. The representative circles of two distinct fractions are either tangent at one point or wholly external to one another.

Theorem 2 Ford circles and the Farey sequence. Suppose that \( h_1/k_1, h_2/k_2, \) and \( h_3/k_3 \) are three consecutive terms in some Farey sequence \( F_n \). Then the circles \( C(h_1, k_1) \) and \( C(h_2, k_2) \) are tangent at

\[
\alpha_1 = \left( \frac{h_2}{k_2} - \frac{k_1}{k_2(k_2^2 + k_1^2)}, \frac{1}{k_2^2 + k_1^2} \right),
\]

and the circles \( C(h_2, k_2) \) and \( C(h_3, k_3) \) are tangent at

\[
\alpha_2 = \left( \frac{h_2}{k_2} + \frac{k_3}{k_2(k_2^2 + k_3^2)}, \frac{1}{k_2^2 + k_3^2} \right).
\]

Moreover, \( \alpha_1 \) lies on the semicircle with diameter \( h_2/k_2 - h_1/k_1 \), and \( \alpha_2 \) lies on the semicircle with diameter \( h_3/k_3 - h_2/k_2 \).

Theorem 3 Largest Ford circle between tangent Ford circles. Suppose that \( C(a, b) \) and \( C(c, d) \) are tangent Ford circles. Then the largest Ford circle between them is \( C(a + c, b + d) \), the Ford circle associated with the mediant fraction.