Probability puzzles

You should be reminded that all random choices (unless otherwise specified) are such that all possibilities are equally likely, and different choices within the same context are by default independent.

(1) **Emperor’s Prisoner:**

    You are a prisoner sentenced to death. The Emperor offers you a chance to live by playing a simple game. He gives you 50 black marbles, 50 white marbles and 2 empty bowls. He then says, 'Divide these 100 marbles into these 2 bowls. You can divide them any way you like as long as you use all the marbles. Then I will blindfold you and mix the bowls around. You then can choose one bowl and remove ONE marble. If the marble is WHITE you will live, but if the marble is BLACK... you will die.

    How do you divide the marbles up so that you have the greatest probability of choosing a WHITE marble?

**The Proportionality Principle:** If various alternatives are equally likely, and then some event is observed, the updated probabilities for the alternatives are proportional to the probabilities that the observed event would have occurred under those alternatives. In mathematical terms, the Proportionality Principle is a re-statement of Bayes’ Theorem. Questions (2) - (5) below can be solved by using this principle.

(2) **Monty Hall problem:**

    Suppose you are on a game show, and you are given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what is behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

(3) **Bertrand’s box:**

    Suppose there are three desks, each with two drawers. One desk contains a gold medal in each drawer, one contains a silver medal in each drawer, and one contains one of each, but you don’t know which desk is which. The question is this: If you open a drawer and find a gold medal, what are the chances that the other drawer in that desk also contains gold?

    Hint: This comes down to figuring out the probability that you’ve picked the gold-gold desk instead of the gold-silver desk.

(4) Your new neighbours have two children of unknown gender. From older to younger, they are equally likely to be girl-girl, girl-boy, boy-girl, or boy-boy. One day you catch a glimpse of a child through their window, and you see that it is a girl. What is the probability that their other child is also a girl?

(5) There is a degenerative disease called Zostritis that 10% of men in a certain population may suffer from in old age. However, if treatments are started before symptoms appear, the degenerative effects can largely be controlled. Fortunately, there is a test that can detect latent Zostritis before any degenerative symptoms appear. The test is not perfect, however:

    * If a man has latent Zostritis, there is a 10% chance that the test will say he does not. (These are called false negatives.)
    * If a man does not have latent Zostritis, there is a 30% chance that the test will say he does. (These are false positives.)

    A random man is tested for latent Zostritis. If the test is positive, then what is the probability that the man has latent Zostritis?
(6) **The Birthday Paradox:**

How big is the probability that two people in a group of 23 randomly chosen people have their birthday on the same day? We will ignore February 29th for the purposes of the problem.

(7) **P. Winkler:**

One hundred people line up to board an airplane. Each has a boarding pass with an assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in his assigned seat?

(8) A person's birthday occurs on a day $i$ with probability $p_i$, where $i = 1, \ldots, n$. (Of course, $p_1 + \ldots + p_n = 1$.) Assume independent assignment of birthdays among different people. In a room with $k$ people, let $P_k = P_k(p_1, \ldots, p_n)$ be the probability that no two persons share a birthday. Show that this probability is maximized when all birthdays are equally likely: $p_i = 1/n$ for all $i$.

(9) **D. Knuth:**

Mr. Smith works on the 13th floor of a 15 floor building. The only elevator moves continuously through floor 1, 2, \ldots, 15, 14, \ldots, 2, 1, 2, \ldots, except that it stops on a floor on which the button has been pressed. Assume that time spent loading and unloading passengers is very small compared to the travelling time. Mr. Smith complains that at 5pm, when he wants to go home, the elevator almost always goes up when it stops on his floor. What is the explanation? Now assume that the building has $n$ elevators, which move independently. Compute the proportion of time the first elevator on Mr. Smith’s floor moves up.

(10) **E. Berlekamp: Betting on the World Series**

You are a broker; your job is to accommodate your client’s wishes without placing any of your personal capital at risk. Your client wishes to place an even $1,000 bet on the outcome of the World Series, which is a baseball contest decided in favour of whichever of two teams first wins 4 games. That is, the client deposits his $1,000 with you in advance of the series. At the end of the series he must receive from you either $2,000 if his team wins, or nothing if his team loses. No market exists for bets on the entire world series. However, you can place even bets, in any amount, on each game individually. What is your strategy for placing bets on the individual games in order to achieve the cumulative result demanded by your client?