Poppy and his Pizza Apparatus (PA)

Poppy’s Pizza Parlor sells combination pizzas. Customers have a choice of \( n \) toppings on their combination pizza.

Poppy is an inventor and has designed an automated pizza-producing machine that he calls the Pizza Apparatus, or PA, for short. The diagram below shows the PA works. After the pizza dough with sauce and cheese is placed on the conveyor belt, the pizza moves beneath the topping dispensers. The dispensers automatically spray toppings on the pizza as it passes by. At the end of the conveyor belt, the pizza emerges ready to bake.

Important notes:

1. The number of topping dispensers in a PA is configurable (from 0 to \( n \)).
2. As a pizza passes under a dispenser, the topping in it is always added to the pizza, i.e. no topping is skipped over.

Pizza Apparatus (PA)
Version 1

1. How many PA’s does Poppy need in order to produce any combination pizza that a customer might order (for all problems, assume that Poppy offers \( n \) toppings from which to choose)?

Two PA’s might produce the same pizza. For example, in the diagram above, if the first dispenser had olives and the last pepperoni, the PA would produce the same pizza.

2. Let \( 0 \leq k \leq n \). For a specific pizza with \( k \) toppings, how many PA’s produce that pizza?

3. Let \( N_k \) be the number of combination pizzas with \( k \) toppings (out of the \( n \) toppings). What value of \( k \) maximizes \( N_k \)?
The Pizza Producing Apparatus (P²A)
Once Poppy discovered how many different PA’s it would take to produce all combination pizzas, he decided to make improvements to his PA in order to reduce the number of PA’s he needed. His first improvement is shown below. It includes diverters that direct the pizzas off the conveyor belt before they reach the end. When the diverter is in the down position, the pizza is pushed off the conveyor belt. In this manner, a single PA can produce as many different combination pizzas as it has dispensers.

Pizza Apparatus (PA)
First Improvement

Upon analyzing how to configure the improved PA’s to produce all combination pizzas, Poppy discovered that he would have many PA’s that had the same toppings in the first few dispensers, so his second improvement (illustrated below) allows the first dispenser to dispense more than one topping (up to \( n \) toppings). He calls his improved PA a Pizza Producing Apparatus or P²A for short.

Pizza Producing Apparatus (P²A)
Second Improvement

The first dispenser may contain multiple toppings
Important note: In a P²A, the diverters come after topping dispensers. Thus in order to produce a pizza with no toppings (cheese only), the first dispenser must be empty.

Since a P²A can produce more than one type of combination pizza depending on which diverter is in the down position, we would like to have a name for two pizzas that can be produced by the same P²A. We will say two pizzas are comparable if there is some P²A that produces both pizzas and incomparable if no P²A produces both of them.

For example, a ham, salami, and onion pizza is comparable to an onion, green pepper, anchovy, ham, and salami pizza. On the other hand, a pepperoni and mushroom pizza and a pepperoni, ham, and onions pizza are incomparable.

4. Poppy is adding “Specials” page to his menu that lists incomparable pizzas (no two pizzas on the page are comparable). Let \( M \) be the maximum number of pizzas that Poppy can list on the Specials page. What is \( M \)?

5. Poppy must have at least \( M \) P²A’s in order to produce all possible combination pizzas (why?). Can Poppy configure \( M \) P²A’s to efficiently produce all combination pizzas? By “efficiently”, we mean that no two P²A’s produce the same pizza.

The Perfect Pizza Producing Apparatus (P³A)

Poppy’s final improvement to P²A is to allow any dispenser to hold multiple toppings. This apparatus was so perfect that Poppy named it the Perfect Pizza Producing Apparatus or P³A for short.

Perfect Pizza Producing Apparatus (P³A)

6. Poppy has reduced the number of pizzas he sells and now offers a limited selection of pizzas. As before, the “Specials” page lists incomparable pizzas. He has determined that the maximum number of pizzas from his limited selection that he can list on the Specials page is \( m \). He will need at least \( m \) P³A’s to produce all the pizzas in his limited selection (why?). Can Poppy configure \( m \) P³A’s that will efficiently produce all of the pizzas he now offers?
Some Non-Pizza Problems

7. Consider a list of 100 people that may include persons who lived in the past or are alive now. Some on the list may be an ancestor of others on the list, i.e. the parent of, or the grandparent of, or the great-grandparent of,.... Show that among the people on the list, either
   a. there is a group of 12 people such that no one in the group is the ancestor of another in the group, or
   b. there is a group of 10 people (call them A, B, C,..., J) such that A is an ancestor of B, B an ancestor of C, C an ancestor of D,....

8. Show that in every sequence of 10 distinct numbers, there is an increasing subsequence of four numbers or a decreasing subsequence of four numbers. For example, in the sequence
   31, 44, 20, 53, 6, 33, 17, 47, 28, 39
   the subsequence 6, 17, 28, 39 is an increasing subsequence of four numbers.
   (Hint: for the sequence $a_1, a_2, a_3,..., a_{10}$ use Dilworth’s Theorem with the partial order “≤”, where $a_i ≤ a_j$ if $a_i ≤ a_j$ and $i ≤ j$. Check that this is a partial order. With this order, what is a chain? an antichain?)

9. Let $D_{600}$ be the poset of positive integer divisors of 600, where $m ≤ n$ if $m$ is a divisor of $n$.
   a. Label the points in the Hasse diagram for $D_{600}$ (below).
   b. Let $r$ be the rank function for $D_{600}$. What is the $r(600)$? Find $N_k$, the number of elements of rank $k$, for $k < r(600)$.
   c. Find a symmetric chain decomposition for $D_{600}$.
   d. Find a maximum size antichain in $D_{600}$.