1 Announcements

Go to http://math.stanford.edu/circle to see all the announcements. The signup sheet for the AMC 10 and AMC 12 contests is linked there, too!

2 Warmup to the Warmup Problems

Problem 1  At a movie theater, the manager announces that a free ticket will be given to the first person in line whose birthday is the same as someone in line who has already bought a ticket. You have the option of getting in line at any time. Assuming that you don’t know anyone else’s birthday, and that birthdays are uniformly distributed throughout a 365-day year, what position in line gives you the best chance of being the first duplicate birthday?

Problem 2  There are 29 trees in a long row. Your pet squirrel is hiding in one of them, out of sight in the top branches. Once each minute, you pick one of the trees (any one of them) and climb it. If he’s there, you find him.

If he’s NOT there, you climb down. As you do, your squirrel leaps from the trees he is in to an adjacent tree (He has two choices unless he’s at one of the trees at the end of the row, in which case he must move to the only available tree) and hides again. In the next minute, you again may pick any one of the trees and climb it, repeating the process. After each unsuccessful search, your squirrel moves to an adjacent tree. (He never stays put!)

If you keep trying, and use a good strategy, can you be certain to eventually find the squirrel? Can you find him in 29 tries? in 30 tries? in 100 tries? or can he keep evading you forever no matter what?

Problem 3  1. (Putnam 2004) Basketball star Shanille OKeals team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

3 Warmup Problems from previous weeks

Problem 4 (asked about in email) Take any positive integer $N$ and consider the set of integers from 1 to $2N$; Partition the set into two subsets, each of size $N$, let’s call them sets $A$ and $B$. Arrange the first set in increasing order, $a_1 < a_2 < a_3 < \ldots < a_N$ and the second set in decreasing order $b_1 > b_2 > b_3 > \ldots > b_N$ and consider the sum

$$|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + \ldots + |a_N - b_N|$$

So, for example, with $N = 5$, you might divide the numbers \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} into two sets of size 5 in this way: set $A$ could be \{1, 4, 5, 7, 9\} and set $B$ might be \{2, 3, 6, 8, 10\}

set $A$ in increasing order is 1,4,5,7,9 and set $B$ in decreasing order is 10,8,6,3,2 so the sum to consider is

$$|1 - 10| + |4 - 8| + |5 - 6| + |7 - 3| + |9 - 2| = 9 + 4 + 1 + 4 + 7 = 25$$
Your task, is to figure out how to partition the integers from 1 to $2N$ into two subsets of size $N$, $A$ and $B$ in such a way that this sum is maximized (and, similarly, how to do it in such a way as to minimize the sum). You could start by trying it for small values of $N$, but ultimately, you’d like to figure out what’s going on for any positive integer $N$.

**Problem 5 (PUTNAM 2012)** (there was actually a very similar problem on the 2012 USAMO). We discussed this in December, but did not complete it. Let $d_1, d_2, ..., d_{12}$ be real numbers in the interval $(1, 12)$. Show that there exist distinct indices $i, j, k$ such that $d_i, d_j, d_k$ are the side lengths of an acute triangle.

**Problem 6 (PUTNAM 2009)** Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}$$

**Problem 7 (2004 Missouri Collegiate Mathematics Competition)** A chess position possesses the following property: On every vertical column and on every horizontal row, there is an odd number of pieces. Prove that there is an even number of pieces on black squares.

**Problem 8** (1983 AIME) A machine shop cutting tool is in the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of $AB$ is 6 cm, and that of $BC$ is 2 cm. The angle $ABC$ is a right angle. Find the square of the distance (in centimeters) from $B$ to the center of the circle.

**Problem 9** The 15-puzzle consists of 15 numbered squares that can slide vertically or horizontally in a $4 \times 4$ square.

Show that it is not possible to rearrange the squares in the inverse order (from 15 to 1).