1. Use the continued fraction technique to find the smallest solution of \( x^2 - 63y^2 = 1 \).

2. Use the continued fraction technique to find the smallest solution of \( x^2 - 75y^2 = 1 \).

3. Find four solutions in positive integers to the equation
\[
x^2 - 5y^2 = 1.
\]

4. How many positive integer solutions are there to the equation \( x^2 - y^2 = 1 \)?

5. Why do we require the \( d \) in the Pell equation \( x^2 - dy^2 = 1 \) to be a non-square positive integer?

6. Find the following remarkable continued fractions:
   
   \begin{align*}
   &\text{(a) } \frac{4}{\pi} \\
   &\text{(b) } e \\
   &\text{(c) } \frac{e^{1/\pi} + 1}{e^{1/\pi} - 1} \\
   &\text{(d) } \left( \sqrt{\phi + 2} - \sqrt{\phi} \right) e^{2\pi/5}, \text{ where } \phi = \frac{1 + \sqrt{5}}{2} \text{ is the golden ratio.}
   \end{align*}

7. Let \((x_k, y_k)\) be the solutions to \( x^2 - 2y^2 = 1 \), as described in the Square-Triangular Number Theorem.

   \begin{enumerate}
   \item Find \( a, b, c, d \) such that
   \[
   x_{k+1} = ax_k + by_k \text{ and } y_{k+1} = cx_k + dy_k.
   \]
   \item Find \( e, f, g, h \) such that if \((m, n)\) satisfies \( n^2 = \frac{m(m+1)}{2} \), then
   \[
   (1 + em + fn, 1 + gm + hn)
   \]
   also produces a square-triangular number.
   \item If \( L \) is a square-triangular number, show that
   \[
   1 + 17L + 6\sqrt{L + 8L^2}
   \]
   is the next largest square-triangular number.
   \end{enumerate}

8. Let \( ST_n \) denote the \( n \)-th square-triangular number. Show that
\[
ST_n = 34ST_{n-1} - ST_{n-2} + 2.
\]

9. What can you say about the size of the \( n \)-th square-triangular number as a function of \( n \)? Study the ratio \( r_n = x_n/y_n \) as \( n \) becomes large. Can you explain your observation?
10. Recall that the general formula for the $n$-th pentagonal number is $P_n = \frac{n(3n-1)}{2}$.

(a) Are there any pentagonal numbers (other than 1) that are also triangular numbers? Are there infinitely many?

(b) Are there any pentagonal numbers (other than 1) that are also square numbers? Are there infinitely many?

(c) Are there any numbers (other than 1) that are simultaneously triangular, square, and pentagonal numbers? Are there infinitely many?

(d) Are there any numbers (other than 1) that are both pentagonal and hexagonal? Are there infinitely many?

11. **Solutions of Pell Equations.**

(a) Suppose that $(x_1, y_1)$ is a solution of the Pell equation $x^2 - dy^2 = 1$. Square both sides of

$$1 = x_1^2 - dy_1^2 = (x_1 + y_1 \sqrt{d})(x_1 - y_1 \sqrt{d})$$

to show that $(x_1^2 + y_1^2d, 2x_1y_1)$ is also a solution. Thus, if we find one solution of $x^2 - dy^2 = 1$, then we can find infinitely many solutions.

(b) The smallest solution of $x^2 - 15y^2 = 1$ is $(4, 1)$. Find two more solutions of this Pell equation.

(c) The smallest solution of $x^2 - 22y^2 = 1$ is $(197, 42)$. Find a solution of this Pell equation whose $x$ is larger than $10^6$.

(d) Prove, using the technique that we used for the Pell equation $x^2 - 2y^2 = 1$, that every solution of the Pell equation $x^2 - 11y^2 = 1$ is of the form

$$x_k + y_k \sqrt{11} = (10 + 3\sqrt{11})^k, \quad k = 1, 2, 3, \ldots.$$

**Note:** Although we know that once we find one solution of Pell’s equation we can find infinitely many solutions, there's currently no known pattern for the size of the smallest solution to $x^2 - dy^2 = 1$. For example, the smallest solution of $x^2 - 61y^2 = 1$ is $(1766319049, 226153980)$, while the smallest solution of $x^2 - 63y^2 = 1$ is $(8, 1)$, and the smallest solution of $x^2 - 65y^2 = 1$ is $(129, 16)$. The smallest solution of $x^2 - 73y^2 = 1$ is $(2281249, 267000)$, while the smallest solution of $x^2 - 75y^2 = 1$ is $(26, 3)$.

12. In 1657, Fermat challenged mathematicians to solve $x^2 - 109y^2 = 1$, $x^2 - 149y^2 = 1$, and $x^2 - 433y^2 = 1$. Can you solve these equations?

13. Investigate the Archimedes cattle problem, which Archimedes (287-212 BC) communicated to students at Alexandria in a letter to Eratosthenes. What’s the statement of the problem? Can you determine the size of the 8 unknowns, and thus the size of the herd?