Warm-up Problems. Choose a few of these problems to work on as you get settled in today. Once you’ve thought about a problem on your own, talk to someone sitting near you about your ideas. More challenging problems are marked with (⋆).

1. Find a two-digit number, the sum of whose digits does not change when the number is multiplied by any one-digit number.

2. Find a way to cut a $3 \times 9$ rectangle into 8 squares.

3. Which integers have the property that if the final digit is deleted, the integer is divisible by the new number?

4. There are three piles of stones: one with 10 stones, one with 15 stones, and one with 20 stones. At each turn, a player can choose one of the piles and divide it into two smaller piles. The loser is the player who cannot do this. Who will win (first or second player), and how?

5. A magic square of order $n$ is an $n \times n$ array containing the integers 1 through $n^2$ such that each number appears only once and the numbers across each row, down each column, and along each main diagonal sum to the same number (called the magic constant). Construct a magic square of order 3. What is the number in the middle cell? Now prove that for any magic square of order 3, you must always have this number in the central cell.

6. Can you arrange the numbers 1, 2, . . . , 9 along a circle in such a way that the sum of two neighbors is never divisible by 3, 5, or 7?

7. Arrange eight of the nine numbers

   $2, 3, 4, 7, 10, 11, 12, 13, 15$

in the vacant squares of the $3 \times 4$ grid shown below in such a way that the arithmetic mean (average) of the numbers in each row and in each column is the same integer.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td></td>
</tr>
</tbody>
</table>

8. How many zeros are there at the end of $103!$? Remember that $n! = n(n − 1)(n − 2) \cdots 3 \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. 

Stanford Math Circle 1 Warm-up Problems
9. Prove that the equation
\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} = 1 \]
has no solutions with \(a, b, c, d, e, f\) all odd natural numbers.

10. Two of the squares of a 7 × 7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are called “equivalent” if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

11. Find the greatest \(n\) for which \(12^n\) evenly divides \(20!\).

12. Let \(P_1, P_2, \ldots, P_{13}\) be thirteen points in the plane. Draw the segments
\[ P_1P_2, P_2P_3, \ldots, P_{12}P_{13}, P_{13}P_1. \]
Is it possible to draw a straight line which passes through the interior of each of these segments?

13. (∗) Each of eight boxes contains six balls. Each ball has been colored with one of \(n\) colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine the smallest integer \(n\) for which this is possible.

14. (∗) Assign to each point \((x, y)\) in the plane a real number in the following way: for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

15. (∗) Let \(a > b > c > d\) be positive integers and suppose
\[ ac + bd = (b + d + a - c)(b + d - a + c). \]
Prove that \(ab + cd\) is not prime.