Warm-up Problems. Choose a few of these problems to work on as you get settled in today. Once you’ve thought about a problem on your own, talk to someone sitting near you about your ideas. More challenging problems are marked with (⋆).

1. Solve the following “cross-number” puzzle. Each entry is one of the integers 0,1,2,3,4,5,6,7,8,9. The clues are as follows. (Note that no answer can begin with 0.)

   ![Cross-number puzzle](image)

   **Across.**

   1. An integer multiple of 8.
   4. A product of the form $1 \times 2 \times 3 \times \cdots \times n$, where $n$ is a positive integer.
   5. A product of two or more consecutive prime numbers.

   **Down.**

   1. An integer multiple of 11.
   2. A value of $2^n$, where $n$ is a positive integer.
   3. An integer multiple of 11.

   **Hint:** There are only two possibilities for 4 Across: $5! = 120$ or $6! = 720$. There are only three possibilities for 2 Down: $128$, $256$, or $512$.

2. An $n \times n$ magic square is a square grid filled with the numbers $1, 2, \ldots, n^2$ such that the sum of the numbers on the main diagonals, in each row, and in each column, are all the same. Complete the following $5 \times 5$ magic square.

<table>
<thead>
<tr>
<th>1</th>
<th>19</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>3</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>
**Hint:** Begin by finding the missing number in the 4th column. What must the 2 missing numbers in the last row sum to? What are the possibilities? What must the 2 missing numbers in the first column sum to? What are the possibilities?

3. Five different numbers are given. By computing all of the different sums of 2 numbers, we get the list

\[ 8, 11, 13, 14, 15, 16, 18, 19, 21, \]

where, possibly, some of the numbers in the list have occurred more than once. Find the 5 numbers.

**Hint:** Call the numbers \(x < y < z < v < w\). What is \(x + y\)? What is \(x + z\)? \(v + w\)? \(z + w\)? Explain why \(x \leq 3\) and \(x \geq 3\).

4. In the following multiplication problem, \(A, B, C, D, E\) are different positive integers. Determine their values.

\[
\begin{array}{cccc}
A & B & C & D & E \\
\times & 4 & & & \\
\hline
E & D & C & B & A
\end{array}
\]

**Hint:** First observe that \(A\) must be equal to 1 or 2. How does \(4E\) relate to \(A\)? Is \(A\) even or odd? Next find \(E\), and finally solve for \(B, C, D\). You’ll end up with \(A = 2, E = 8, B = 1, C = 9, D = 7\).

5. Find the smallest integer whose first digit is 7 and which is reduced to \(1/3\) of its original value when its first digit is transferred to the end. Then find all integers with this property.

**Hint:** Explain why the last digit of such a number must be 1. Then find the next-to-last digit. Continue, moving right to left, until you obtain the digit 7.

6. The positive integers 1,2,3,4,5,6,7,8,9 are arranged along the sides and corners of a triangle, as illustrated. Find an arrangement so that the sums of the squares of the four positive integers along each side are equal.

![Triangle with numbers](image)

**Hint:** Let the sum of the squares be \(t\). Let the corners be \(a, b, c\), and the other numbers \(d, e, f, g, h, i\). Write the three equations for the sums of the squares, and then add them. Show that \(a^2 + b^2 + c^2\) must be a multiple of 3. Finally, show that \(t\) must be 126.

7. Find and prove the rule suggested by the examples below.

\[
\begin{align*}
1 &= 1 \\
2 + 3 + 4 &= 1 + 8 \\
5 + 6 + 7 + 8 + 9 &= 8 + 27 \\
10 + 11 + 12 + 13 + 14 + 15 + 16 &= 27 + 64
\end{align*}
\]

**Hint:** \((n^2 + 1) + (n^2 + 2) + \cdots + (n + 1)^2 = n^3 + (n + 1)^3\). To prove this, observe that the terms on the left-hand side are in arithmetic progression.
8. Find and prove a formula for the value of

\[ 1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + 4! \cdot 4 + \cdots + n! \cdot n. \]

**Hint:** \( \sum_{k=1}^{n} k \cdot k = (n+1)! - 1 \). You can prove this by induction, or by observing that \( (n+1)! - n! = n!(n+1) - n! = n! \cdot n \).

9. A fraction processing machine takes a fraction \( f \) and produces a new fraction \( \frac{1-f}{1+f} \). If a fraction \( \frac{p}{q} \) is fed into the machine, what fraction will be produced after 2010 processes?

**Hint:** What happens if you put in \( p/q \)? Then put the result into the machine. What do you observe?

10. In equilateral triangle \( ABC \), the point \( P \) is on \( AB \) so that \( AP = AB/3 \) and the point \( Q \) is on \( BC \) so that \( BQ = BC/3 \), and the point \( R \) is on \( CA \) so that \( CR = CA/3 \). The lines \( CP, AQ, BR \) enclose a triangle. Find the ratio of the area of this triangle to the area of \( ABC \).

**Answer:** The ratio is 1/7.

11. \((*)\) Show that there are infinitely many integers \( n \) such that \( n, n+1, n+2 \) are each the sum of the squares of two integers. (For example, \( n = 0 \) is such an integer since \( 0^2 + 0^2 = 1 \).

12. \((*)\) The octagon \( P_1P_2P_3P_4P_5P_6P_7P_8 \) is inscribed in a circle, with the vertices around the circumference of the circle in the given order. The polygon \( P_1P_3P_5P_7 \) is a square of area 5, and the polygon \( P_2P_4P_6P_8 \) is a rectangle of area 4. Find the maximum possible area of the octagon.

**Answer:** The maximum possible area is \( 3\sqrt{5} \).

13. \((*)\) Is there a positive integer \( n \) such that \( n \) has exactly 2000 prime divisors and \( 2^n + 1 \) is divisible by \( n \)?