Today’s problems were all based on congruences, mostly mod 7. We also played around with digital numbers at the beginning, which are nine-digit numbers containing each digit from 1 to 9 exactly once. Here is a compilation of some of the questions which appeared during the math circle, along with a few extras for good measure.

1. How many digital numbers are there? Prove that they are all divisible by 9.
2. What is the probability that a digital number is divisible by 7? What is the probability that it is of the form $9p$, where $p$ is a prime?
3. Can you find two digital numbers whose sum is also a digital number?
4. In theory a digital number could be twice another digital number, or three times another, all the way up to eight times as big. Which multiples actually occur?
5. What day of the week will it be 7654 days from today? (Today is Sunday, Aviv.)
6. Use the technique we developed to determine the day of the week on which the following famous dates fell: July 20, 1969 (first man to walk on the moon), May 25, 1977 (release date of original Star Wars), and September 5, 1997 (Mother Theresa died).
7. Perform the following computations in mod 7 arithmetic. (In other words, fill in the blank with a number from 0 to 6 to create a true statement.)

\[
\begin{align*}
1 + 2 + 3 + \cdots + 10 &\equiv \underline{\quad} \mod 7, \\
40^5 &\equiv \underline{\quad} \mod 7 \\
234 \cdot 345 &\equiv \underline{\quad} \mod 7, \\
5 \div 6 &\equiv \underline{\quad} \mod 7.
\end{align*}
\]

8. Determine the value of the quantity $10 \cdot 20 \cdot 30 \cdot 40 \cdot 50 \cdot 60 \mod 7$ in two different ways. First reduce each number separately before multiplying. Then go back and factor out all the 10’s instead before combining what is left. In this way explain why $10^6 \equiv 1 \mod 7$. (This illustrates the method by which Fermat’s Little Theorem is proved.)

9. In a certain sequence each number is the product of all previous numbers, plus 5. If the first number is 4, then show that none of the terms is divisible by 7.

10. Suppose that a positive integer is divisible by 7. If you remove the last digit, double it, then subtract this from what is left of the original number, then explain why the result will also be divisible by 7.

11. Find the remainder when $(102^{73} + 55)^{37}$ is divided by 111. (Hint: calculate the remainder when it is divided by 3 or 37, then figure out how to combine your results.)

12. (Hard!) Let $p \geq 5$ be a prime. Prove that when the sum

\[
\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}
\]

is combined into a single fraction, the resulting numerator is divisible by $p^2$. 