This time we investigated two seemingly unrelated topics. First, we defined a suave number as a positive integer which can be written as the sum of two perfect squares. Thus \( n \) is suave if and only if we can write \( n = a^2 + b^2 \) for integers \( a \) and \( b \) not both zero. The first several suave numbers are 1, 2, 4, 5, 8, 9, and 10. (Note that I’ve decided that 0 is not suave! I think the theory is neater this way.) Next we considered Gaussian integers. These are complex numbers of the form \( a + bi \), where \( i^2 = -1 \) and \( a \) and \( b \) are integers. Thus \( 3 - 2i \), \( 7i \), and \( 4 \) are all Gaussian integers. We then discovered that the two topics are actually very closely related, as suggested by the following exercises.

1. List the suave numbers from 1 to 50. Confirm the following observations: the product of two suave numbers is also suave, but three times a suave number is not suave.

2. Prove that if \( n \) is suave, then \( 3n \) is not suave. (Idea: suppose there are such suave numbers, and let \( n \) be the smallest. Thus we can write \( n = a^2 + b^2 \) and \( 3n = c^2 + d^2 \). Now explain how to find a smaller example by proving that both \( c \) and \( d \) must be divisible by 3.)

3. Let \( \alpha = 10 - 11i \) and \( \beta = 3 - 2i \). Compute \( \alpha + \beta \), \( \alpha - \beta \), \( \alpha \cdot \beta \), and \( \alpha/\beta \).

4. We define the norm of a Gaussian integer \( \alpha = a + bi \) as \( N(\alpha) = a^2 + b^2 \). (The norm measures how “large” \( \alpha \) is, in some sense.) Determine \( N(2 + 5i) \) and \( N(1 - i) \). Now multiply \( (2 + 5i)(1 - i) \), and compute the norm of the result. What do you notice? (In fact, it is true in general that \( N(\alpha \beta) = N(\alpha)N(\beta) \).)

5. Find all Gaussian integers \( \alpha \) for which \( N(\alpha) = 1 \). (There are four of them, called the units—every Gaussian integer is divisible by these four numbers.)

6. Now find all \( \alpha \) such that \( N(\alpha) = 13 \). Then try \( N(\alpha) = 11 \) and \( N(\alpha) = 25 \). (There are twelve answers for the latter question. Don’t forget that \( N(5i) = 25 \), for example.)

7. Find a way to write \( 2 + 9i \) as a product of two non-trivial factors. In other words, don’t use the numbers 1, \(-1\), \( i \), or \(-i \), which always divide any Gaussian integer. (Hint: use the norm.) Now do the same thing for 29.

8. Show that \( 4 + i \) and \( 11 \) are prime Gaussian integers. (Hint: use the force. I mean norm.)

9. Now suppose that \( p \equiv 1 \mod 4 \). We know that \( 1 \cdot 2 \cdot 3 \cdots (p-2)(p-1) \equiv -1 \mod p \) by Wilson’s theorem. Use this fact to argue that \( \left( \frac{p-1}{2} \right)^2 + 1 \) is a multiple of \( p \).

10. It is a fact that if a prime number divides \( \alpha \beta \), then it must divide either \( \alpha \) or \( \beta \). The previous questions implies that there is some number \( N \) such that \( N^2 + 1 \) is divisible by \( p \). In other words, \( p \) divides \( (N + i)(N - i) \). Explain why neither factor is divisible by \( p \). Conclude that \( p \) is not a prime in the Gaussian integers.

11. We just saw that if \( p \equiv 1 \mod 4 \), then \( p \) is not prime in the Gaussian integers. Thus we can write \( p = \alpha \beta \) for non-trivial numbers \( \alpha \) and \( \beta \). Explain why this means that \( N(\alpha) = p \), and then deduce that \( p = a^2 + b^2 \). Thus \( p \) is suave! And you’re done.