Today we continue our quest to prove geometric results by applying algebra to complex numbers. Recall that once we establish the correspondence between points in the plane and complex numbers, we can then interpret various arithmetic operations in geometric terms. (See the previous worksheet for further details.) In this problem set we will develop some of the techniques needed to deal with equilateral triangles, as opposed to the squares which occupied our attention last time.

1. To begin, we define \( \omega \) to be the complex number obtained by rotating the point corresponding to 1 by 120° about the origin. By using what you know about 30°–60°–90° triangles, show that \( \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \).

2. It is a fact that multiplying a complex number by \( \omega \) has the effect of rotating that number by 120° about the origin. (Just as multiplying by \( i \) rotated numbers by 90°.) Use this fact to explain why \( \omega^3 = 1 \).

3. Calculate the following quantities by any means: \( 1 + \omega + \omega^2 \), \( (1 + \omega)(1 + \omega^2) \), and \( \omega^8 \). Then explain why multiplying by \( -\omega^2 \) corresponds to a rotation by 60°.

4. Let \( A, B, \) and \( C \) be complex numbers situated at the vertices of an equilateral triangle. Demonstrate that \( C = A - \omega^2(B - A) \), assuming that the points occur in counterclockwise order around the triangle. (Note that this expression reduces to just \( C = -\omega A - \omega^2 B \).) What is the expression for \( C \) if the points are listed in clockwise order instead?

5. Let \( ABDC \) be a parallelogram, and construct equilateral triangles \( ABE \) and \( BDF \) on sides \( AB \) and \( BD \) that are located exterior to the parallelogram. (Remember from last week that we can let \( A = 0 \), in which case \( D = B + C \) because \( ABDC \) is a parallelogram.) Prove that triangle \( CEF \) is also equilateral.

6. Now let \( ABC \) be an arbitrary triangle in the plane. Construct equilateral triangles \( BCD \), \( ACE \), and \( ABF \) on sides \( BC \), \( AC \), and \( AB \), located exterior to the original triangle. Prove that segments \( AD \), \( BE \), and \( CF \) have the same length and cross one another at 60° angles.

7. It is well-known that the centroid of a triangle is located two-thirds of the way from any vertex to the midpoint of the opposite side. Use this fact to show that the centroid \( G \) of triangle \( ABC \) is the number \( G = \frac{1}{3}(A + B + C) \).

8. Let \( P, Q \), and \( R \) be the centroids of equilateral triangles \( BCD \), \( ACE \), and \( ABF \) in the problem above. Prove that triangle \( PQR \) is itself equilateral. (Triangle \( PQR \) is known as the outer Napoleon triangle.)

9. Explain how we may also conclude (without any further computations!) that if the three equilateral triangles in the previous problem are constructed internally, so that they overlap the original triangle, then the centroids still form an equilateral triangle. (Known as the inner Napoleon triangle.) Finally, show that the centroids of both Napoleon triangles coincide.