Let’s imagine that we introduce a new coin system. Instead of using pennies, nickels, dimes, and quarters, let’s say we agree on using only 4-cent and 7-cent coins. One might point out the following flaw of this new system: certain amounts cannot be exchanged, for example, 1, 2, or 5 cents. On the other hand, this deficiency makes our new coin system more interesting than the old one, because we can ask the question: “which amounts can be changed?” We will see shortly that there are only finitely many integer amounts that cannot be exchanged using our new coin system. A natural question, first tackled by Ferdinand Georg Frobenius and James Joseph Sylvester in the nineteenth century, is: “what is the largest amount that cannot be exchanged?” As mathematicians, we like to keep questions as general as possible, and so we ask: given coins of denominations \(a\) and \(b\)—positive integers without a common factor—can you give a formula \(g(a, b)\) for the largest amount that cannot be exchanged using the coins \(a\) and \(b\)? This problem and its generalization for coins \(a_1, a_2, \ldots, a_n\) is known as the Frobenius coin-exchange problem. To study the Frobenius number \(g(a, b)\), we use the Euclidean Algorithm. For integers \(a\) and \(b\) that have no common factor, this algorithm yields integers \(x\) and \(y\) such that \(ax + by = 1\).

**Problems**

1. Find \(g(4, 7)\) and \(g(5, 11)\).

2. Find \(x\) and \(y\) such that \(4x + 7y = 1\). Find another \(x\) and \(y\) such that \(4x + 7y = 1\).

3. Find \(x\) and \(y\) such that \(5x + 11y = 1\). Find \(x\) and \(y\) such that \(5x + 11y = 39\).

4. Show that, if \(t\) is a given integer, we can always find integers \(x\) and \(y\) such that \(4x + 7y = t\). Generalize to two coins \(a\) and \(b\) with no common factor.

5. Show that, if \(t\) is a given integer, we can always find integers \(x\) and \(y\) such that \(4x + 7y = t\) and \(0 \leq x \leq 6\). Generalize to two coins \(a\) and \(b\) with no common factor.

6. Show that the following recipe for determining whether or not a given amount \(t\) can be changed (using the coins 4 and 7) works: Given \(t\), find integers \(x\) and \(y\) such that \(4x + 7y = t\) and \(0 \leq x \leq 6\). Then \(t\) can be changed precisely if \(y \geq 0\). Generalize to two coins \(a\) and \(b\) with no common factor.

7. Use the previous argument to re-compute \(g(4, 7)\). Generalize your argument to compute \(g(a, b)\), for any two coins \(a\) and \(b\) with no common factor.

8. Suppose \(t\) is an integer between 1 and \(ab - 1\) that is not a multiple of \(a\) or \(b\). Prove that if the amount \(t\) can be changed then \(ab - t\) cannot be changed, and conversely, if \(t\) cannot be changed then \(ab - t\) can be changed.

9. Prove that there are \(\frac{1}{2}(a - 1)(b - 1)\) amounts that cannot be changed.

10. Think about why \(g(a, b)\) actually exists, if \(a\) and \(b\) have no common factor. More generally, prove that the general Frobenius problem is well defined. That is, show that, given \(a_1, a_2, \ldots, a_d\) with no common factor, every sufficiently large integer is representable (in terms of \(a_1, a_2, \ldots, a_d\)).
11. Next week we will study the counting sequence

\[ r_k = \# \{(m, n) \in \mathbb{Z}^2 : m, n \geq 0, ma + nb = k\} \].

In words, \( r_k \) counts the representations of \( k \in \mathbb{Z}_{\geq 0} \) as nonnegative linear combinations of \( a \) and \( b \). The Frobenius problem asks for the largest among the \( r_k \)'s that is 0. Prove that \( r_{k+ab} = r_k + 1 \).

**A few remarks**

The simple-looking formula for \( g(a, b) \) that you have found in \((\)\) inspired a great deal of research into formulas for the general Frobenius number \( g(a_1, a_2, \ldots, a_d) \), with limited success: While it is safe to assume that the formula for \( g(a, b) \) has been known for more than a century, no analogous formula exists for \( d \geq 3 \). The case \( d = 3 \) is solved algorithmically, i.e., there are efficient algorithms to compute \( g(a, b, c) \) [2, 4, 5], and in form of a semi-explicit formula [3, 7]. The Frobenius problem for fixed \( d \geq 4 \) has been proved to be computationally feasible [1, 6], but not even an efficient practical algorithm for \( d = 4 \) is known. The formula in \((\)\) is due to Sylvester and was published as a math problem in the *Educational Times* more than a century ago [9]. For more on the Frobenius problem, we refer to the research monograph [8]; it includes more than 400 references to articles written about the Frobenius problem.

**References**


Matthias Beck  
math.sfsu.edu/beck