Complex Numbers

Some History

Let us try to solve the equation \( x^3 = 15x + 4 \).

\( x = 4 \) is an obvious solution.

Also, \( x^3 - 15x - 4 = (x - 4)(x^2 + 4x + 1) = 0 \) yields 2 more solutions

\( x = -2 \pm \sqrt{4 - 1} = -2 \pm \sqrt{3} \).

If we draw the graphs of \( y = x^3 \) and \( y = 15x + 4 \), we can see that they intersect at 3 points.

Notice also that any equation \( x^3 = 3px + 2q \) must have at least one solution!

Yet the Cardan-Tartaglia formula (*Ars Magna*, 1545)

\[
x = \frac{1}{3} \left[ q + \sqrt{q^2 - p^3} \right] + \frac{1}{3} \left[ q - \sqrt{q^2 - p^3} \right]
\]

gives

\[
x = \frac{1}{3} \left( \sqrt{2 + \sqrt{-121}} + \sqrt{2 - \sqrt{-121}} \right).
\]

Around 1572, Bombelli guessed that \( \sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1} \),

\( \sqrt[3]{2 - \sqrt{-121}} = a - b\sqrt{-1} \), and found \( a = 2, \ b = 1 \). Then

\[
x = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4.
\]

And thus the complex numbers really entered the scene of mathematics in 1572.

But they remained somewhat “unreal” until Gauss (almost two and a half centuries later) introduced the idea of treating them as points on the plane.
**Definition** A complex number is a point \( z = (x, y) \) in the Cartesian plane in which the \( x \)-axis is measured in ordinary (real) units and the \( y \)-axis is measured in a different (imaginary) unit \( i \). The real number \( x \) is called the real part of \( z \) \((x = \text{Re}(z))\), and the real number \( y \) is called the imaginary part \((y = \text{Im}(z))\). (Note: Both the real and the imaginary parts of \( z \) are real numbers.)

![Diagram of complex number](image)

**Operations on Complex Numbers**

**Algebra**

<table>
<thead>
<tr>
<th>Vector</th>
<th>( z = (x, y), \ w = (s, t) )</th>
</tr>
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<tbody>
<tr>
<td>Addition:</td>
<td>( z + w = (x + s, y + t) )</td>
</tr>
</tbody>
</table>

**Geometry**

- **Scalar**
  - **Multiplication:** \( z = (x, y), \ k \text{ real} \)
  - \( kz = (kx, ky) \)

We can now think of complex numbers as vectors. More precisely, the set of complex numbers can be thought of as equivalent to the real vector space \( \mathbb{R}^2 \).

**Corollary:** \( z = (x, y) = x(1, 0) + y(0, 1) = x + iy \) (Cartesian form of \( z \)).

**Question:** What is \( z - w \)? (Algebraically and geometrically).
Important Notions

**Algebra**

**Modulus:** \(|z| = \sqrt{x^2 + y^2}\)

**Conjugate:** \(\overline{z} = x - iy\)

**Questions:**
1. What is \(|z - w|\)? (Algebraically and geometrically).
2. Is \(|\overline{z}| = |z|\)?

**Geometry**

Complex Multiplication

We want \(i^2 = -1\), and we also want the usual algebraic rules to still hold: associative, commutative laws for both addition and multiplication, and also, the distributive law. So, if \(z = x + iy\), then \(iz = i(x + iy) = -y + ix\).

**Question:** Geometrically, how can we get \(iz\)?

From the picture above, what can you conclude about the relationship of the vectors \(z\) and \(iz\)?

In general, how can we get \(wz\), where \(w = s + it\)?
Let \( P, Q, M, N, R, T \) denote the points of the complex plain that correspond to the points \( z, w, sz, tz, itz, wz = (s + it)z = sz + t(iz) \), respectively.

\( ROMT \) is a parallelogram, and since \( \angle RON \) is right, \( ROMT \) is a rectangle. Consider the triangles \( TMO \) and \( QXO \) where \( X \) corresponds to the number \( s = s + i0 \). 

\[
\frac{TM}{OM} = \frac{OR}{OM} = \frac{|itz|}{|sz|} = \frac{|iz|}{|s|} = \frac{OX}{OX},
\]
and hence \( \triangle TMO \sim \triangle QXO \).

Thus
\[
\angle TOM = \angle QOX, \quad \text{and}
\]
\[
\angle TMO = \angle QXO.
\]

Since \( MT = |xz| = |s||z| = OX|z| \), so \( wz = OT = OQ|z| = |w||z| \).

So to get \( wz \), we rotate \( z \) counterclockwise through the angle \( \angle QOX \), then stretch it by a factor of \( |w| \).

### Another Form of Complex Numbers

Complex numbers are points of a plane, so we need 2 coordinates to denote ( = to locate) them. But we can use different ways to choose these coordinates. In particular, we can use polar coordinates: \( z = (r, \theta) \), where \( r = |z| \), and \( \theta = \arg(z) \mod 2\pi \).

\[
z = r(\cos \theta + i \sin \theta)
\]

Notice: We have seen that if \( z = (|z|, \theta), \quad w = (|w|, \mu), \) then

\[
z w = (|z||w|, \theta + \mu).
\]

Hence \( |zw| = |z||w| \) (multiply moduli);

\[
\arg(zw) = \arg(z) + \arg(w) \quad \text{(add arguments)}.
\]

<table>
<thead>
<tr>
<th>Cartesian Form</th>
<th>Polar Form</th>
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<tbody>
<tr>
<td>( z = x + iy ), ( w = s + it )</td>
<td>( z =</td>
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<tr>
<td>( z =</td>
<td>z</td>
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<tr>
<td>( zw = (xs - yt) + i(ys + xt) )</td>
<td>( zw =</td>
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<tr>
<td>(</td>
<td>zw</td>
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Problems
1. Use polar form to show that every complex number $z \neq 0$ has two square roots. Hint: What happens (geometrically!) when a complex number is squared? Therefore, what is necessary to “unsquare” (that is to take the square root) of a number (geometrically)?

2. In Cartesian form, one square root of $x + iy$ when $y \geq 0$ is
$$\sqrt{x + iy} = \sqrt{\frac{x^2 + y^2 + x}{2}} + i\sqrt{\frac{x^2 + y^2 - x}{2}}.$$ Verify this. What is the other root? What if $y < 0$?

3. Find all square roots of these complex numbers:
   (a) $-1$  
   (b) $-4$  
   (c) $i$  
   (d) $2i$  
   (e) $-i$  
   (f) $-1 - i\sqrt{3}$  
   (g) $2 + 2i$  
   (h) $4$

Properties of conjugates and moduli

1. $|z + w| \leq |z| + |w|$ (Triangle Inequality)
2. $|zw| = |z||w|$
3. $\overline{w} = |w|^2$
4. $\overline{zw} = \overline{z}\overline{w}$
5. $z + w = \overline{z} + \overline{w}$, and $\overline{z - w} = \overline{z} - \overline{w}$
6. $\left| \frac{z}{w} \right| = \left| \frac{\overline{z}}{\overline{w}} \right|$
7. $\left( \frac{z}{w} \right) = \left( \frac{\overline{z}}{\overline{w}} \right)$

Notice:
$$\frac{1}{w} = \frac{\overline{w}}{|w|^2} = \frac{\overline{w}}{|w|^2} \in \mathbb{C} \quad \text{(if w \neq 0)}$$

Fundamental Theorem of Algebra

(Proved in 1799 by Gauss, 21 years old at the time).

A polynomial equation of degree $n > 1$ with complex coefficients has a solution in the set of all complex numbers, $\mathbb{C}$. (And thus $\mathbb{C}$ is algebraically closed).
Corollary: A polynomial equation over \( \mathbb{C} \) of degree \( n \) has exactly \( n \) roots in \( \mathbb{C} \), counting multiplicities.

Some Other Important Properties

1. \( z^n = |z|^n (\cos n\theta + i \sin n\theta) \) (where \( z = |z|(\cos \theta + i \sin \theta) \)) (De Moivre’s Theorem).

2. This theorem can be used to find all \( n^{th} \) roots of complex numbers.

Examples: Find all solutions of the given equations.

1. \( z^3 = 8 \)
2. \( z^5 = 1 \)

ADDITIONAL EXERCISES AND PROBLEMS

1. Perform the indicated operations, and reduce each of the following numbers to the form \( x + iy \), where \( x, y \in \mathbb{R} \).
   
   (a) \( (1 - i)(2 - i)(3 - i) \) 
   (b) \( (\sqrt{3} + i)^6 \) 
   (c) \( \frac{4 + 3i}{3 - 4i} \) 
   (d) \( \frac{5 - z}{5 + z} \), where \( z = 4 + 3i \)

2. Find necessary and sufficient conditions for the three numbers \( z_1, z_2, \) and \( z_3 \) to be the vertices of an equilateral triangle.

3. Let \( z = a + ib, \ a, b \in \mathbb{R} \). Find conditions on \( a \) and \( b \) such that
   (a) \( z^4 \) is real
   (b) \( z^4 \) is purely imaginary.

4. Find the absolute value (modulus) of
   (a) \( 3 + 2i \) 
   (b) \( -1 + i\sqrt{3} \) 
   (c) \( -i(1 + i)(2 - 3i)(4 + 3i) \) 
   (d) \( \frac{3 - i}{(-1 + 2i)(2 - 3i)} \)

5. If \( a, b, \) and \( n \) are positive integers, prove that there exist integers \( x \) and \( y \) such that \( (a^2 + b^2)^n = x^2 + y^2 \).
6. (a) Show that if \( z \) is a root of a polynomial equation with real coefficients (i.e., all the coefficients are real numbers), then \( \bar{z} \) is also a root.
   (b) Show that a polynomial equation with real coefficients and odd degree must have at least one real root.

7. Solve the equations.
   (a) \( z^3 - i = 0 \)
   (b) \( z^4 + 1 = 0 \)
   (c) \( z^5 + 32 = 0 \)
   (d) \( z^6 - 1 = 0 \)

8. Find the smallest positive integers \( m \) and \( n \) satisfying \( (1 + i\sqrt{3})^m = (1 - i)^n \).

9. Suppose that \( A_1A_2...A_n \) is a regular polygon inscribed in a circle of radius \( r \) and center \( O \). Let \( P \) be a point on \( OA_i \) extended beyond \( A_i \). Show that
   \[
   \prod_{k=1}^{n} PA_k = OP^n - r^n.
   \]

10. Given a point \( P \) on the circumference of a unit circle and the vertices \( A_1, A_2, ..., A_n \) of an inscribed regular polygon of \( n \) sides, prove that \( PA_1^2 + PA_2^2 + ... + PA_n^2 \) is a constant.

11. Let \( A_0, A_1, A_2, A_3, A_4 \) divide a unit circle into five equal parts. Prove that the chords \( A_0A_1 \) and \( A_0A_2 \) satisfy the equation \( (A_0A_1)(A_0A_2))^2 = 5 \).

12. (a) If \( z_1 + z_2 + z_3 = 0 \), and \( |z_1| = |z_2| = |z_3| = 1 \), show that these three complex numbers are the vertices of an equilateral triangle inscribed in the unit circle.
   (c) Can you extend the previous result to the case of four complex numbers? A \( n \) complex numbers?