You should read the book *The Shape of Space* by Jeff Weeks. It’s a great book. Some of these ideas are from that book.

1 Gluing Diagrams Using Squares

![Gluing Diagrams Using Squares](image)

2 Gluing Diagrams Using Hexagons

What surfaces do these gluing diagrams represent?

![Gluing Diagrams Using Hexagons](image)

How many other essentially different gluing diagrams can you make by identifying the sides of a hexagon in pairs? How many different topological surfaces can you get in this way?
3 Euler Number

\[ \chi = V - E + F \]

1. A soccer ball is a sphere made out of pentagons and hexagons arranged so that three shapes meet around each vertex. Prove that a soccer ball must contain 12 pentagons.

2. How many hexagons does a soccer ball have? Is it possible to make a topological sphere using only pentagons and hexagons, with three faces around each vertex, with a different number of hexagons?

3. Use Euler number to prove that it is not possible to put 5 points on a plane and connect each pair of points with a line segment in such a way that the lines don’t cross.

4. Use Euler number to prove that, on a plane, it is not possible to connect each of 3 houses with each of 3 utilities in such a way that the lines don’t cross.

4 More Surfaces

5. What surface do you get when you cut a disk out of a projective plane? (A disk is the inside of a circle.)

6. Use gluing diagrams to show that the connected sum of two projective planes is a Klein bottle.

5 Planet Casson

Planet Casson is entirely dry land, except for a system of canals. One canal runs around the equator, and three canals run from three different spots on the equator to join up at the north pole. (So the canals form a shape like the edges of a tetrahedron.) The canals separate the land into 4 land masses. There is one ferry boat for each land mass that circles its land mass in a counterclockwise direction, traveling along the bounding canals. Unfortunately, the canals are too narrow for two ferries to pass each other, and there have been many crashes. Can you devise a ferry schedule with no crashes? (Ferries are not allowed to travel backwards.)

What if you use a different arrangement of canals ... for example, an arrangement in the shape of the edges of a cube, or an octahedron?

What if Planet Casson is shaped like a torus?