1. Recall that complex numbers are numbers of the form $a + ib$, where $a$ and $b$ are real numbers and $i = \sqrt{-1}$ is the complex unit. The Fourier transform depends on certain important complex numbers called “roots of unity”. Let’s discover some of their properties, as well as a useful way of writing arbitrary complex numbers.

(a) Compute the values of $(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^2$ and $(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^3$

**Solution**

Products of complex numbers can be computed by using FOIL and then simplifying, remembering that $i^2 = -1$. So

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \cdot \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) \cdot \frac{\sqrt{3}}{2}i + \frac{\sqrt{3}}{2}i \cdot \left(-\frac{1}{2}\right) + \frac{\sqrt{3}}{2}i \cdot \frac{\sqrt{3}}{2}i$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{4}i - \frac{\sqrt{3}}{4}i + \frac{3}{4}i^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Similarly then, we can write $(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^3$ as

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 \cdot \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \cdot \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right),$$

so the product is of the form $(a - b)(a + b) = a^2 - b^2$ with $a = -\frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}i$. Thus the product is $\frac{1}{4} - (\frac{3}{4}i^2) = 1$.

(b) What are all complex numbers $z$ such that $z^4 = 1$? (How do you know you found them all?) Plot them all on the complex plane, and find their sum.

**Solution**

Clearly 1 has this property, and also $-1$. But then, since $i^2 = -1$, we also know that $i^4 = (i^2)^2 = 1$. Since changing the sign of a number and then taking it to an even power doesn’t change the result, we also have that $-i$ has fourth power equal to 1. So that’s four complex numbers with the desired property.

In fact, we can see that this is all of them, because these are actually “roots” of the polynomial $z^4 - 1$, that is, numbers which solve the equation $z^4 - 1 = 0$. Since the highest exponent of $z$ in this polynomial is 4, it can have at most four roots, so we got them all.

(c) Where on the complex plane are the solutions of $z^{10} = 1$? What are the complex coordinates of these points?
Solution
Plotting the points on the last problem makes a diamond or a circle. In fact, in general the numbers which have $z^n = 1$ for some $n$ are just $n$ evenly spaced points around the unit circle, starting at 1. For $n = 10$, this means that the points can be written in terms of trig functions:

$$z_i = \cos(\theta) + i \cdot \sin(\theta), \quad \theta = 0, \left( \frac{2\pi}{10} \right), 2 \cdot \left( \frac{2\pi}{10} \right), \ldots, 9 \cdot \left( \frac{2\pi}{10} \right).$$

(Remember that $i \cdot \left( \frac{2\pi}{10} \right)$ is the angle which is $i$ tenths of a full circle, in radians.)

(d) The **n-th roots of unity** are the complex solutions $z$ to the equation $z^n = 1$. What is the sum of the $n$-th roots of unity?

Solution
Trying a couple of examples (the first two problems computed the 3rd and 4th roots of unity), it looks like the sum should be 0. In fact, this is true for all numbers $n$. This is easy to see for $n$ an even number, since the roots come in pairs of opposite numbers which cancel with each other. This argument doesn’t work for $n$ an odd number though! One way to see it in this case is by thinking of the numbers as vectors and seeing what their sum means geometrically. Another way we’ll see in a problem below.

(e) We define the **complex exponential**, or the exponential function of a complex number by the formula

$$e^{a+ib} := e^a (\cos(b) + i \sin(b)).$$

The notation uses polar coordinates to represent a point in the complex plane: $e^{a+ib}$ gives the point which is at an angle of $b$ radians from the real axis and is a distance of $e^a$ away from the origin:

Show that complex exponential satisfies the usual property for products of exponentials. That is, for $z = a + ib$ and $w = c + id$, show that $e^z e^w = e^{z+w}$.

Solution
Let’s write it out in terms of the definitions and see what happens. We have

$$e^z e^w = e^{a+ib} e^{c+id} = e^a (\cos(b) + i \sin(b)) e^c (\cos(d) + i \sin(d))$$

$$= e^a e^c (\cos(b) + i \sin(b)) (\cos(d) + i \sin(d))$$

$$= e^{a+c} \left( (\cos(b) \cos(d) - \sin(b) \sin(d)) + i(\sin(b) \cos(d) + \cos(b) \sin(d)) \right)$$

$$= e^{a+c} (\cos(b + d) + i \sin(b + d)) = e^{a+c+i(b+d)} = e^{z+w}$$
Notice that we needed to use the angle sum formulas for sin and cos to finish the argument.

(f) What does multiplication of complex numbers mean geometrically for the corresponding points on the complex plane?

**Solution**
This is one of the coolest things about complex numbers. If you look at complex numbers, as in the above diagram, as vectors with a length from the origin and an angle from the positive real axis, then multiplying them corresponds to multiplying their lengths and adding their angles. This follows from the multiplication formula we worked out in the previous example.

(g) Using complex exponential notation, how can we write the $n$-th roots of unity? Does this lead to a simpler way to find their sum?

**Solution**
The $n$-th roots of unity can be written in polar coordinates as

$$z_i = e^{i\theta}, \quad \theta = 0, \left(\frac{2\pi}{n}\right), 2 \cdot \left(\frac{2\pi}{n}\right), \ldots, (n-1) \cdot \left(\frac{2\pi}{n}\right).$$

But another way of writing them is as powers of $z_1$, namely, since we are adding the angle $2\pi/n$ each time, $z_i$ is just the product of $z_1$ with itself $i$ times. That means that the sum of the $n$-th roots of unity can also be written as

$$S = 1 + z_1 + z_1^2 + z_1^3 + \cdots + z_1^{n-1}$$

This is just a geometric series! Since the starting term is 1, the ratio is $z_1$, and the highest power is $n-1$, that means the sum is given by the geometric series formula by

$$S = \frac{z_1^n - 1}{z_1 - 1}$$

But since $z_1$ is an $n$-th root of unity, its $n$-th power is 1, so the fraction is just zero.
2. A function $f$ on the integers is called **periodic with period** $N$ if for any $n$ we have $f(n+N) = f(n)$. Such a function has the same value at any two numbers in the same congruence class modulo $N$, so it can be described by a list of its values on the integers $0$ through $N-1$. We will write, for instance, the list $(1, 0, 3\pi, 2, \sqrt{2})$ to describe a periodic function $f$ with period $5$ and values

\[
\begin{array}{c|ccccccc}
  n & \cdots & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
  f(n) & \cdots & \sqrt{2} & 1 & 0 & 3\pi & 2 & \sqrt{2} & 1 & 0 \\
\end{array}
\]

We will explore an alternative way of looking at periodic functions.

(a) We can write $f(n) = (-1)^n$ for a function which alternates back and forth between $1$ and $-1$ (periodic with period $N = 2$), and $f(n) = i^n$ for a function which repeats the pattern $(1, i, -1, -i)$ (periodic with period $N = 4$). In a similar fashion, how can we use roots of unity to write a function which repeats the pattern $(1, 0, -1, 0)$?

**Solution**

For a problem like this, sometimes it’s best to just get your hands dirty and try some examples! In this case, we want something that has zero in the second and fourth positions, so adding two functions together which have opposite values at those positions might work. In fact, by adding together $f_1(n) = i^n$ and $f_2(n) = (-i)^n$, we end up with the function $(2, 0, -2, 0)$, so the function we want is

\[
g(n) = \frac{i^n + (-i)^n}{2} = \frac{1}{2} i^n + \frac{1}{2} (-i)^n.
\]

(b) How about the pattern $(0, 1, 0, -1)$?

**Solution**

We see that this function is just the previous one, but translated to the right by one, so now we can that function to more easily come to a solution. We want $h(n) = g(n - 1)$, which we can then write as

\[
h(n) = g(n - 1) = \frac{1}{2} i^{n-1} + \frac{1}{2} (-i)^{n-1} = \frac{1}{2i} i^n - \frac{1}{2i} (-i)^n.
\]

(c) How about the pattern $(1, 1, -1, -1)$?

**Solution**

Again we should try to use what we already know to make our task simpler. Notice that when you add two functions together, if one function has value zero at a point, then the sum has the value of the other function. In our case, the sum of $g$ and $h$ gives us the function we’re looking for, so

\[
g(n) + h(n) = \left(\frac{1}{2} i^n + \frac{1}{2} (-i)^n\right) + \left(\frac{1}{2i} i^n - \frac{1}{2i} (-i)^n\right) = \frac{1-i}{2} i^n + \frac{1+i}{2} (-i)^n
\]
(d) Now let’s make a general construction for period $N = 4$ functions. How can we write functions following the patterns $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$? Using these, how can we write an arbitrary period 4 function, following, say, the pattern $(a_0, a_1, a_2, a_3)$?

**Solution**

Here we put together a couple of nice pieces of the puzzle. First, the basic building block is the periodic function $(1, 0, 0, 0)$. In order to find a representation for this function in terms of powers of $1, i, -1$ and $-i$, remember the basic fact from problem 1: The sum of the $n$-th roots of unity is zero when $n$ is at least 2. In particular, notice that:

$$1^n + i^n + (-1)^n + (-i)^n$$

...gives you $1 + i + (-1) + (-i)$ when $n = 1$, $1 + (-1) + 1 + (-1)$ when $n = 2$, and $1 + (-i) + (-1) + i$ when $n = 3$—all of which are just sums of roots of unity, and so are just zero. The odd ball out is when $n = 0$, which obviously just gives us $1 + 1 + 1 + 1 = 4$.

But this isn’t a problem; since everything else is zero, we can just divide by 4 to get the function we want:

$$u_0(n) = \frac{1}{4} + \frac{1}{4}i^n + \frac{1}{4}(-1)^n + \frac{1}{4}(-i)^n = \sum_{k=0}^{3} \frac{i^{kn}}{4}.$$

Like in part (b) above, we can translate our first solution to get the remaining functions:

$$u_1(n) = \sum_{k=0}^{3} \frac{i^{k(n-1)}}{4} = \sum_{k=0}^{3} \frac{i^{kn}}{4i^k} = \frac{1}{4} + \frac{1}{4}i^n - \frac{1}{4}(-1)^n - \frac{1}{4}(-i)^n$$

$$u_2(n) = \sum_{k=0}^{3} \frac{i^{k(n-2)}}{4} = \sum_{k=0}^{3} \frac{i^{kn}}{4i^{2k}} = \frac{1}{4} - \frac{1}{4}i^n + \frac{1}{4}(-1)^n - \frac{1}{4}(-i)^n$$

$$u_3(n) = \sum_{k=0}^{3} \frac{i^{k(n-3)}}{4} = \sum_{k=0}^{3} \frac{i^{kn}}{4i^{3k}} = \frac{1}{4} - \frac{1}{4}i^n - \frac{1}{4}(-1)^n + \frac{1}{4}(-i)^n$$

Then similarly to part (c), we can get the general function $(a_0, a_1, a_2, a_3)$ by adding multiples of $u_0, u_1, u_2, u_3$ to get $f(n) = a_0u_0(n) + a_1u_1(n) + a_2u_2(n) + a_3u_3(n)$. Then writing this in sigma notation and changing the order of the sum (notice below that $j$ changes position with $k$) we can find the actual coefficients for each root of unity:

$$f(n) = a_0u_0(n) + a_1u_1(n) + a_2u_2(n) + a_3u_3(n)$$

$$= \sum_{j=0}^{3} a_j u_j(n)$$

$$= \sum_{j=0}^{3} a_j \left( \sum_{k=0}^{3} \frac{1}{4i^j} i^{kn} \right)$$

$$= \sum_{k=0}^{3} \left( \sum_{j=0}^{3} \frac{a_j}{i^j} \right) i^{kn}$$
So the coefficient of $i^{kn} = (i^k)^n$ in the general representation is

$$
\hat{a}_k = \frac{1}{4} \sum_{j=0}^{3} a_j \cdot i^{-jk},
$$

so that $f(n) = \hat{a}_0 + \hat{a}_1 i^n + \hat{a}_2 (-1)^n + \hat{a}_3 (-i)^n$.

(e) Based on this, can you propose a general method of representing a periodic function on the integers in terms of roots of unity?

**Solution**

In fact, this formula can easily be extended to functions with an arbitrary period $N$ by replacing any 3s or 4s with $N - 1$s or $N$s, and by replacing the 4th root of unity $i$ by the $N$th root of unity $e^{2\pi i/N}$. This gives us a general way to write the coefficients for the $N$-periodic function $(a_0, a_1, a_2, \ldots, a_{N-1})$:

$$
\hat{a}_k = \frac{1}{N} \sum_{j=0}^{N-1} a_j \cdot e^{-(2\pi i/N)jk}
$$

3. We define the **Fourier transform**\(^1\) of a periodic function $f$ with period $N$ as the new periodic function $g$ whose values are the coefficients of $f$ written in terms of roots of unity. We denote the Fourier transform of $f$ by $\hat{f}$. (Can two different functions with the same period have the same Fourier transform?)

We also can define a sort of multiplication operation on periodic functions with the same period $N$. If $f$ and $g$ are $N$-periodic, define a new function $f \ast g$, their **convolution**, by

$$(f \ast g)(n) := \frac{1}{N} \sum_{k=0}^{N-1} f(k)g(n - k).$$

This is a sort of averaging procedure on the functions, and it interacts with the Fourier transform in a very nice way.

(a) If $f$ is the periodic function $(2, 2, 0, 0)$ and $g$ is the periodic function $(1, 2, 3, 4)$, compute $f \ast g$ and $g \ast f$. How are $f \ast g$ and $g \ast f$ related?

**Solution**

We can compute, for instance,

$$f \ast g(1) = \frac{1}{4} (f(0)g(1) + f(1)g(0) + f(2)g(-1) + f(3)g(-2)) = \frac{1}{4} (2 \cdot 2 + 2 \cdot 1 + 0 \cdot 4 + 0 \cdot 3) = 3/2,$$

and

$$g \ast f(1) = \frac{1}{4} (g(0)f(1) + g(1)f(0) + g(2)f(-1) + g(3)f(-2)) = \frac{1}{4} (1 \cdot 2 + 2 \cdot 2 + 3 \cdot 0 + 4 \cdot 0) = 3/2.$$

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\(^1\)Named after 18th century French mathematician Joseph Fourier, who studied questions about heat transfer and vibrations
The computations for other values are similar, and the result is that $f \ast g = g \ast f$, and their common value is the periodic function $(\frac{1}{2}, \frac{3}{3}, \frac{5}{2}, \frac{7}{2})$.

(b) For an arbitrary $N$-periodic function $f$, if $g$ is periodic with values $(N/2, N/2, 0, 0, \ldots, 0)$, how is the convolution $f \ast g$ related to $f$? What if $g$ has values $(N, 0, 0, \ldots, 0)$?

**Solution**

For an $N$-periodic function $f$ with values $(a_0, a_1, a_2, \ldots, a_{N-1})$, the convolution comes out to be

$$f \ast g(n) = \left( \frac{a_0 + a_1}{2}, \frac{a_1 + a_2}{2}, \ldots, \frac{a_{N-2} + a_{N-1}}{2}, \frac{a_{N-1} + a_0}{2} \right).$$

In other words, this convolution gives you the periodic function whose terms are the averages of consecutive terms in the original function.

(c) Let $f$ be 4-periodic with values $(1, 0, -1, 0)$ and $g$ be 4-periodic with values $(6, 6i, -6, -6i)$. Compute the function $\hat{f} \ast \hat{g}$. How does this function relate to $\hat{f}$ and $\hat{g}$?

**Solution**

To do these comparisons, we need to compute four things: $f \ast g$, $\hat{f} \ast \hat{g}$, $\hat{f}$, and $\hat{g}$. You should do this computation! Practice with examples is important to understanding any concept. Once you’ve done the computation, make sure that you got the correct results, which are:

$$f \ast g = (3, 3i, -3, -3i)$$

$$\hat{f} \ast \hat{g} = (0, 3, 0, 0)$$

$$\hat{f} = (0, \frac{1}{2}, 0, \frac{1}{2})$$

$$\hat{g} = (0, 6, 0, 0)$$

In particular, it looks like $\hat{f} \ast \hat{g}$ is just the periodic function that you get by multiplying together the corresponding values of $\hat{f}$ and $\hat{g}$, i.e. $\hat{f} \ast \hat{g} = (0 \cdot 0, \frac{1}{2} \cdot 0, 6 \cdot 0, 0)$.

(d) Propose a general relation between $\hat{f} \ast \hat{g}$ and $\hat{f}$, $\hat{g}$. Can you prove it?

**Solution**

The above observation is in fact a very important general fact. If $\hat{f} = (\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{N-1})$ and $\hat{g} = (\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_{N-1})$, then the Fourier transform of $f \ast g$ is just the component-wise product

$$\hat{f} \ast \hat{g} = (\hat{a}_0 \cdot \hat{b}_0, \hat{a}_1 \cdot \hat{b}_1, \hat{a}_2 \cdot \hat{b}_2, \ldots, \hat{a}_{N-1} \cdot \hat{b}_{N-1}).$$

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The proof involves some tricky manipulations of sigma notation:

\[
\hat{f} * g(k) = \frac{1}{N} \sum_{j=0}^{N-1} f \ast g(j)e^{-\left(\frac{2\pi i}{N}\right)jk}
\]

\[
= \frac{1}{N} \sum_{j=0}^{N-1} \frac{1}{N} \sum_{l=0}^{N-1} f(l)g(j-l)e^{-\left(\frac{2\pi i}{N}\right)jk}
\]

\[
= \frac{1}{N^2} \sum_{l=0}^{N-1} f(l) \sum_{j=0}^{N-1} g(j-l)e^{-\left(\frac{2\pi i}{N}\right)jk}
\]

\[
= \frac{1}{N^2} \sum_{l=0}^{N-1} f(l) \sum_{j=0}^{N-1} g(j)e^{-\left(\frac{2\pi i}{N}\right)(j-l)k}e^{-\left(\frac{2\pi i}{N}\right)lk}
\]

\[
= \frac{1}{N} \sum_{l=0}^{N-1} f(l)e^{-\left(\frac{2\pi i}{N}\right)lk} \left( \frac{1}{N} \sum_{j=0}^{N-1} g(j-l)e^{-\left(\frac{2\pi i}{N}\right)(j-l)k} \right)
\]

\[
= \left( \frac{1}{N} \sum_{l=0}^{N-1} f(l)e^{-\left(\frac{2\pi i}{N}\right)lk} \right) \left( \frac{1}{N} \sum_{j'=0}^{N-1} g(j')e^{-\left(\frac{2\pi i}{N}\right)j'k} \right) = \hat{f}(k)\hat{g}(k)
\]

The two trickiest steps are in the labeled lines. In line (1), we move the inner $1/N$ to the outside of the summation (it doesn’t depend on either of the index variables $j$ or $l$), and we change the order of the summation. Since the bounds of the summations are both independent of the index variables, this is allowed as long as any terms depending on the index variables remain inside the corresponding summations. Since $f(l)$ only depends on $l$ and not $j$, we can move it outside of the $j$ summation expression.

In line (2), we split the exponential term by writing $j = (j-l)+l$, which gives an exponent of $-(2\pi i/N)(j-l)k - (2\pi i/N)l$. Since this is a sum of two numbers, it corresponds to a product of their corresponding exponentials (since $e^{a+b} = e^a \cdot e^b$), and these are the two exponential terms we get in the next line.

In line (3), we “re-index” the summation. Instead of writing the sum in terms of the variable $j$, taking values between 0 and $N - 1$, we write it in terms of a new variable $j' = j - l$. Since $j$ takes values between 0 and $N - 1$, $j'$ takes values between $-l$ and $N - 1 - l$, so the summation turns into

\[
\frac{1}{N} \sum_{j'=0}^{(N-1)-l} g(j')e^{-\left(\frac{2\pi i}{N}\right)j'k}.
\]

However, since $g$ is $N$-periodic, we can replace the terms $j' = -l, -l + 1, \ldots, -1$ with those values plus $N$ and the expression in the sum doesn’t change. That means we can instead write the summation over values of $j'$ between 0 and $N - 1 - l$, along with values of $j'$ between $-l + N$ and $-1 + N$—which gives us back exactly the values of $j'$ between 0 and $N - 1$. If this is unclear, the easiest way to understand what’s happening is to write out all of the terms in both sums and see how they line up.

(e) The periodic function $e$ with values $(N, 0, 0, \ldots, 0)$ is called the **neutral function** for convolution because for any function $f$, $f \ast e = e \ast f = f$, i.e. convolution with $e$ doesn’t
change the function \( f \). Functions \( f \) and \( g \) are called **convolution inverses** of each other if \( f * g = e \). (In analogy with usual numerical multiplication, this is like saying that 2 and 1/2 are inverses of each other because their product is the neutral element 1.) However, not every periodic function has a convolution inverse. Using the Fourier transform, how can we determine whether a function \( f \) has a convolution inverse? Is there a simple way of describing the inverse function?

**Solution**

If we rephrase everything in terms of the Fourier transform, then the property of being convolution inverses comes down to simple multiplication. If we assume that \( f \) and \( g \) are convolution inverses, then take the Fourier transform of the equation \( f * g = e \), we get that \( \hat{f} * \hat{g} = \hat{f}\hat{g} = (1,1,\ldots,1) \). So in other words, the **Fourier transforms of \( f \) and \( g \) multiply to get the constant 1 function**.

This is great, because it gives us an easy way of finding the convolution inverse of a function \( f = (a_0,a_1,\ldots,a_{N-1}) \): First take the Fourier transform, then take the reciprocal of the values of the Fourier transform as the Fourier coefficients of a new function: \( \hat{b}_i = 1/\hat{a}_i \). So if

\[
    f(n) = \sum_{k=0}^{N-1} \hat{a}_k e^{-\frac{2\pi}{N}kn},
\]

then the convolution inverse \( g \) of \( f \) is just

\[
    g(n) = \sum_{k=0}^{N-1} \frac{1}{\hat{a}_k} e^{-\frac{2\pi}{N}kn}.
\]

...As long as this makes sense. In fact, it could be that \( \hat{a}_k = 0 \) for some choice of \( k \), in which case, it doesn’t make sense to take the reciprocal, since this would be dividing by zero. So there’s the condition we need. A periodic function \( f \) has a convolution inverse if and only if all its Fourier coefficients are nonzero, in which case the convolution inverse \( g \) is given by the expression above.